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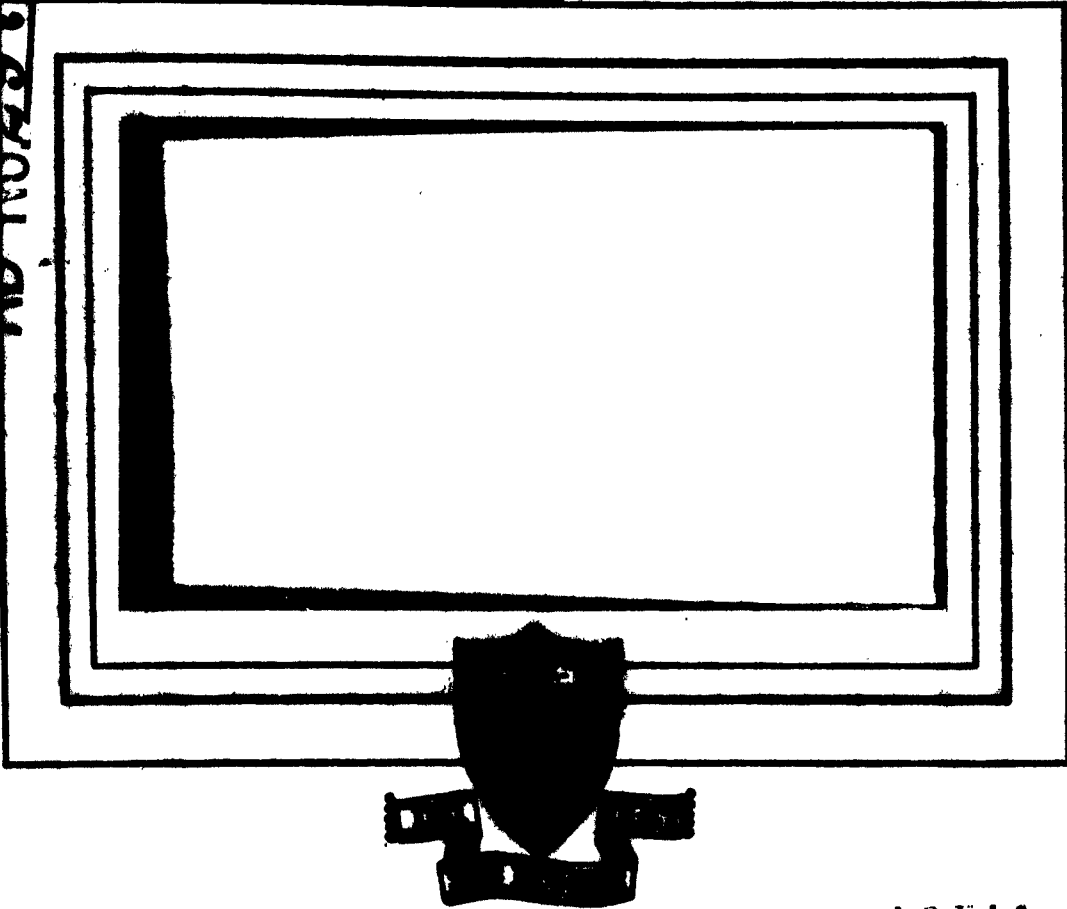
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THE POWER OF A COALITION

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1. Introduction.

Questioning whether a characteristic function adequately represents a game given, say, in a normal form, R. D. Luce and H. Raiffa argue that for non-constant-sum games, there is no a priori reason to believe that whenever a coalition forms, its complement also forms, and that even if the complement forms, it is perfectly possible for the players in the original coalition to receive more than their value, since the cost to the complement in holding the payments down may be excessive. (See [3] pp. 190-192, 203-204).

This criticism does not hold for zero-sum games, at least from a normative point of view:

"For zero-sum games, it can be argued that if a coalition T forms, it can never enforce more than v(T) since the remaining players, who are assumed to be rational and unconstrained by any social limitations, will certainly form the coalition -T". ([3] p. 203).

We would like to question whether a value of a coalition is an adequate measure of its "power" even if a game is given only in characteristic function form. Obviously, in addition to what a coalition can make by itself, it may gain some more "strength" by threatening not to cooperate with other players, thus causing them losses.

This paper is motivated by the results of an experiment, in which high school children were playing 3-person games, all of them with a characteristic function: $v(i) = 0$, $i = 1, 2, 3$, $v(12) = 50$, $v(13) = 50$, $v(23) = 0$, $v(123) = 50$[1]. This game has a unique point in the core, which is (50, 0, 0). This imputation is also the "center" of each of the von Neumann-Morgenstern solutions (see [12]), and it is also the only imputation which occurs in the various bargaining sets (see [1]). At first, the games indeed ended in such a way that player 1 received

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(1) See [5] for details. Actually, the coalition {123} was not allowed, but the students found a way to avoid this restriction (see text). Similar results were obtained by playing the game $v(i) = 0$, $i = 1, 2, 3$, $v(12) = v(13) = 60$, $v(23) = 10$, $v(123)$ not allowed.

almost 50, giving another player a very small amount. This is in good agreement
with the various theories. However,/more games were played, the weak players
realized that it is worthwhile to flip a coin under the condition that the loser
would "go out of the game," thus "forcing" a split 25:25 between the winner and
player 1. Eventually, player 1 realized that he ought to offer player 2 or player
3 some amount around $12\frac{1}{2}$, in order that the "coalition" {23} will not form, because
$12\frac{1}{2}$ was the expectation for each of the weak players. Obviously, the players,
guided by some "justice" feelings, "knew" that the "right thing to do" in a
2-person game $v(1) = v(2) = 0$, $v(12) = 50$, is to split equally, and on the basis
of this knowledge they acted as if the value of the coalition {23} was 25 and not
0.

Our study here is concerned with the question: how to measure a "power"
of a coalition if we know that such or a similar "standard of fairness" exists
among the players.

If we knew, for example, that an outcome of a particular game is "bound
to be" a certain imputation $(x_1, x_2, \ldots, x_n)$, we could claim that the power of
each coalition in this particular game is the sum of the payments to its members.
Of course, in this case the game would have "reduced" to an _inessential_ game,
which is uninteresting for further study.

If we had some a priori estimation, as to what the outcome imputation
"ought to be," we could use this estimation to replace the super-additive charac-
teristic function by an additive one. This in fact was done by L. S. Shapley
in [8] when he offered his value. However, as L. S. Shapley puts it, his value
"is best regarded as an a priori assessment of the situation, based on either
ignorance or disregard of the social organization of the players," while we wish
to incorporate these social organizations in the game.

J. von Neumann and O. Morgenstern say in [12], that a particular out-
come of a game depends on factors not given in the description of the game. They
refer e.g. to the "bargaining abilities" of the players, their actual method of
negotiation, etc. In Sections 2 and 3 we shall describe some of these factors,

which we name "standards of fairness." Some of them are to be determined by experiments, and others by theoretical considerations. Each "standard of fairness" will be described by a Thrall generalized characteristic fucntion (see [11], possessing certain properties, and will be used to determine a "power" for each coalition (Section 4). This will be done by regarding the game as played among the various coalitions, each of which has pure strategies of breaking itself into "pressure groups."

The power will then be considered as a new characteristic function, and we shall find out that it reduces every game to a constant-sum game. This is not too surprising. Once we assume that rational players must come out with an imputation, clearly the more a group of players get, the less the rest of the players will receive. It is true that this feature is not reflected in the usual description of a game, say, via its characteristic function, and tnis is why we require some data on the "psychology" of the players. It is the same kind of data which states that 2 players, confronted with a non-constant-sum game, will split in a certain particular way.

This paper is intended to meet the criticism which questions the possibility of applying n-person Game Theory to real-life situations. The fact that such a criticism is mainly directed to non-constant-sum games suggests that it is not the various solution theories which are at fault, but rather that one ought to look for some further information about the nature of the game and its players, if one wants to arrive at some more convincing applications. This paper suggests that a search for the "standard of fairness" of the players may provide the required information.

In Section 4 we prove that an iteration of the procedure to get from the power of a coalition to a "new" power will not lead to new results. Section 5 indicates a modified version of the "power" of a coalition and a comparison between the two "powers" is made in Section 8.

In Section 6 we slightly modify the bargaining set theory (see R. J. Aumann and M. Maschler [1]), to take account of the requirement that imputations should result in each game.

We illustrate our method in Section 7 by studying in detail the three person game for two possible standards of fairness. We discuss both the von Neumann-Morgenstern solution and the bargaining set of these games, interpreted via the powers of the coalitions, in a particular numerical example.

2. Some heuristic arguments leading to a game space.

We are concerned with a cooperative n-person game, described by a pair $(v; N)$, where $N = (1, 2, \ldots, n)$ is the set of the players and v is the von Neumann-Morgenstern characteristic function of the game. (See [12]). The characteristic function, therefore, is assumed to satisfy

(2.1) $v(\phi) = 0$ (ϕ is the empty coalition),

(2.2) $v(B \cup C) \geq v(B) + v(C)$ whenever B and C are disjoint coalitions.

An imputation in a game $(v; N)$ is an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers, satisfying

(2.3) $x_i \geq v(i)$, $i = 1, 2, \ldots, n$ (Individual rationality requirement),

(2.4) $x_1 + x_2 + \ldots + x_n = v(N)$ (Group rationality requirement).

In this paper we require:

Postulate 1. (J. von Neumann and O. Morgenstern [12]. The outcome of an n-person cooperative game should be an imputation[1].

Thus, a rational player is expected to refuse receiving less than what he can obtain by acting alone, and all the players are expected to form an outcome which yields a maximal total payoff to them.

Given a game $(v; N)$, one may wish to know which coalitions will or should form. The answer is extremely simple if $v(N)$ satisfies the condition

(2.5) $v(N) > v(B) + v(N-B)$,

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(1) Where x_i represents the payment to player i, $i = 1, 2, \ldots, n$.

for each coalition B other than N and the empty coalition. Indeed, by Postulate 1,
the
only /coalition N can form. From this point of view, one may then argue that the

values of all the multi-person coalitions, except N, are <u>irrelevant</u> to the outcome,

since these coalitions will never form. One may suggest a "fair" split such as dividing

equally the difference $v(N) - \{v(1) + v(2) + \ldots + v(n)\}$ among the players, and giving

each player i his value v(i) in addition.

It is pointless to attack such an argument, and, as a matter of fact, any

similar argument, for a mathematical model is always excellent for those people who

are willing to obey its rules. Besides, there are some appealing morals in this

particular model. However, we wish to explore here models in which multi-person

coalitions, other than N, may influence the outcome. Since these coalitions may

not appear in the final outcome,[1] we must allow their appearance in <u>intermediate</u>

<u>stages</u> of the negotiations.

A formation of a coalition in an intermediate stage may influence the

rest of the game in two ways:

    a. The members of such a coalition B assume themselves at an intermediate

       stage an amount v(B).

    b. The formation of such a coalition may <u>reduce</u> the number of players for

       the rest of the game.

<u>Example 2.1</u> Consider a game between three firms, 1, 2 and 3, having the characteristic

function: v(12) = 100, v(123) = 120 and v(B) = 0 otherwise. Assume that forming a

multi-person coalition means a total merging of the involved firms into one firm. It

seems to us that firms 1 and 2 will act wisely, if they form an intermediate coalition,

splitting the profits, say, equally, and later allow the third firm to join them.

Adhering to the above principle[2] of "fair share," the final outcome may well be

(55, 55, 10).

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(1) If (2.5) holds.

(2) This principle comes for illustration purpose only. Later on, we
shall not be committed to it.

Aspect a, above, seems to be described adequately by the characteristic function v, at least inasmuch as the game is given a priori in a characteristic function form; but, apparently, factor b is not reflected by the characteristic function as the above and the following example indicate:

Example 2.2 Let v(12) = 100, v(13) = 100, v(123) = 120 and v(B) = 0 otherwise, be a characteristic function of a 3-person game. The coalition {23}, although of a 0 value, may still act as a one player, and if it does, a final outcome (60, 30, 30) may result. However, if the above principle of "fair share" holds, the coalition {23} will do still better by forming an intermediate coalition, deciding not to listen to any offer from player 1 to one of them but enter the game as two players. In this case the outcome will be (40, 40, 40). We would like to take the stand that if the players are inclined to accept the above principle of "fair share," then any decision of the players 2 and 3 on forming or not forming an intermediate coalition, should be based on the expected profit 80(= 40 + 40) and not on the actual value 0.

Thus, when an intermediate coalition is formed, it may partition itself into subcoalitions, who enter the next stage of the game as single players. These subcoalitions will be called pressure groups. One of our problems will be to find out what pressure groups should form if a coalition forms.

Finally, we want to drop the "fair share" principle. One player may be a firm and another — an employee; or perhaps the game is between two publishers and a writer, or simply between several people having different "bargaining abilities," etc. Sometimes, there even exist "potential" players outside the game, who may tacitly enter the picture[1]. In such situations, and many others, an equal share is not a realistic assumption.

Instead of this principle, we shall allow a variety of possible principles, which will be called generically standards of fairness. Intuitively, they may rep-

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[1] An inventor, if not satisfied, may threaten to look for another investor, even though this other investor does not participate in the negotiations and perhaps is not even known to the players.

resent statements such as: "A writer usually gets 15 percent of the gross sale."
"Profits are divided in the same proportions as the investments." "Wages for cer-
tain occupations are such and such." "Profits for such and such risks should be so
much." These are various accepted rules, by which a society usually determines the
division of profits, prior to the actual bargaining.

Take, for example, a 2-person non-constant-sum game between a writer and
a publisher. Nothing in the characteristic function of this game tells why the
writer should get 15 percent of the gross sale, and for this reason, J. von Neumann
and O. Morgenstern claim that the particular outcome is beyond their theory, and
depends on other factors. Yet, if we knew that the "15 percent" is an accepted
rule in this case, then, the outcome would be determined[1].

Suppose now that the game is between a writer (player 1), a publisher
(player 2) and an agent (player 3). One may think of a case when an imputation will
arise only if the three players cooperate. If the costs are, say, 50 percent of the
expected gross sale, then a "standard of fairness" may dictate the following profits
to the pressure groups:

Partition into pressure groups	Profits % of the gross sale
1, 2, 3	15%, 25%, 10%
12, 3	35%, 5%
13, 2	35%, 15%
23, 1	40%, 10%
123	50%

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(1) One may argue that the 15 percent rule is itself an "equilibrium" out-
come of a bigger game. This may be the case, but since we do not want to carry the
impossible analysis of games involving all the players in the world, we are ready to
accept such rules as given a priori.

Explanation: "Normally," a writer is "entitled" to 15 percent, and the agent to 10 percent of the gross sale.

If players 1 and 2 decide to cooperate, they can point out to the agent that he takes very little responsibility: the publisher already committed himself which proves that he believes in the value of the book. In fact, "all they want" from the agent is that he provides his advertising facilities, for which 5 percent profits are plenty.

If player 1 and player 3 cooperate, they "relieve" the publisher from finding and convincing an agent. "All they want" is to use his printing machines, and since he has "nothing" to lose, he might as well be satisfied with 15 percent. He will "never" be able to make such easy profits at such a small risk.

If 23 is the first pressure group to form, then this is already a "publishing company" with high reputation, and "good connections;" the author is putting his book in safe hands and, in fact, they are "doing him a favor" if they agree to publish his book. He should be more than satisfied if he gets 10 percent from this distinguished institution.

Such arguments cannot be deduced from the description of the game, given either in characteristic function form or even in a normal form, because they specify side payments for various situations. Yet, people use such arguments every day, taking into account social mores and various roles that each pressure group may assume. We, therefore, require that information such as the one given in the above table will be known in addition to the characteristic function.

In the previous example, we omitted, for simplicity considerations which are influenced by the value of each pressure group. Obviously, a rich group and a poor group may have different "bargaining abilities." Sometimes, a rich pressure group assumes a greater responsibility, and therefore demands more. In some other cases, it will be satisfied with less profits because it can afford to lose[1].

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(1) A rich country pays more to the United Nations, even though it has the same rights as a poor country.

Mathematically, a <u>standard of fairness</u> for a game $(v;N)$ is a vector-valued function

(2.6) $$\varphi([P]) \equiv \{\varphi_1([P]),\ \varphi_2([P]),\ \ldots,\ \varphi_m([P])\},$$

defined for each partition $[P] \equiv (P_1,\ P_2,\ \ldots,\ P_m)$ of N into pressure groups, and satisfying:

<u>Postulate 2.</u>

(2.7) $$\varphi_j([P]) \geq v(P_j),\quad j = 1,\ 2,\ \ldots,\ m,$$

(2.8) $$\varphi_1([P]) + \varphi_2([P]) + \ldots + \varphi_m([P]) = v(N).$$

Relation (2.7) states that a <u>rational</u> pressure group will refuse to participate in the final imputation if it is offered less than what it can make by itself. Relation (2.8) simply claims that all the amount $v(N)$ will be shared by the pressure groups which are formed.

Postulate 2 is similar in spirit to Postulate 1, if we regard the pressure groups as players in a "pseudo game." We use this term, because the "players," i.e., the various subsets of N, are not players in the ordinary sense. They may appear and disappear, since no two overlapping pressure groups may participate ($=$form) simultaneously[1].

<u>Definition 2.1</u> The pair $(\varphi([P]);N)$, where N is the set of players in a game $(v;N)$, and $\varphi([P])$ is a standard of fairness, satisfying (2.7) and (2.8), is called a <u>game space</u>.

A game space is, therefore, represented by a Thrall generalized characteristic function (see [11]).

The next section will attempt to convince the reader that a function $\varphi([P])$ can be estimated in many real-life situations; however, it seems appropriate to answer now the following question:

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(1) Also, values of coalitions of pressure groups were not defined.

Suppose we knew the function $\varphi([P])$ for a particular game — to what extent are we short of predicting the actual outcome of this game? The answer is that we still have to determine which intermediate coalitions should arise, what side payments should they offer their members and how each coalition will partition itself into pressure groups. If, say, each player acts as a 1-person intermediate coalition, then the outcome is assume to be $(\varphi_1([P]), \varphi_2([P]), \ldots, \varphi_n([P]))$, where $[P] = (1, 2, \ldots, n)$. But this need not be the outcome if other intermediate coalitions form in the negotiations process. In fact, the main purpose of bargaining is to form such coalitions in a clever way, so that some players may do better than the above imputation. We therefore shift the interest to the study of the coalitions that may form and assume that once we know who formed and how they chose their pressure groups, then the outcome is practically known[1].

3. Possible constructions of a game space.

The author is not an expert in the Social Sciences, and his suggestions given in a, b, and c below are written with the hope that they will encourage experimental and theoretical social scientists to undertake a thorough research in these topics.

We shall indicate here various possible estimations of the standards of fairness thay may occur in real life situations.

a. Pure bargaining.

Conceptually, if a partition $[P] \equiv (P_1, P_2, \ldots, P_m)$ is formed, the pressure groups are asked to determine the splits of $v(N)$, under the assumptions that no other pressure groups will form. As a "first approximation," we take the view that the "bargaining abilities" are independent of the wealth of multi-person coalitions of the pressure groups. Therefore, one may observe how well the players do if one presents the players with the game[2] $(u;[P])$, where

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(1) Except for the various side payments which still are to be determined.

(2) The idea to use the outcomes of (essentially) the game $(u;[P])$ to measure the bargaining abilities of the players is due to L. S. Shapley (see [9]).

$$(3.1) \qquad \qquad u(P_1, P_2, \ldots, P_m) = v(N),$$

$$(3.2) \qquad u(P_{j1}, P_{j2}, \ldots, P_{jk}) = v(P_{j1}) + v(P_{j2}) + \ldots + v(P_{jk}), \quad 1 \leq k < m.$$

Here, j_υ, $\upsilon = 1, 2, \ldots, k$, are pairwise distinct and take values from 1 to m. Other coalitions in this game are flat.

According to J. von Neumann and O. Morgenstern [12], the particular outcome of such a game is determined by the "bargaining abilities of the pressure groups." However, if such bargaining abilities exist, conceivably they can be determined by laboratory experiments. If such experiments show consistency, one may start varying the parameters $v(N)$ and $v(P_j)$, $j = 1, 2, \ldots, m$, and study how the outcome changes. Finally, one has to prove that this method yields a first approximation, if coalitions of the pressure groups are not flat. For example, if a consistent outcome for a 3-person game $u(P_1) = 20$, $u(P_2) = 10$, $u(P_3) = 5$, $u(P_1 P_2 P_3) = 100$, is, say, (40, 40, 20). Will the participants feel "moral obligations" to pay more to the first two players if we allow for a coalition $\{P_1, P_2\}$ with a value[1] 40?

b. **Accepted precedents.**

In many business situations there exist accepted rules for dividing profits. One common method is to divide the profits in the same proportion to the discounted investments. If skills are contributed in a partnership, it is customary to evaluate these skills in terms of money, and refer back to the same rule.

In Section 2, we indicated that wages are relatively fixed. Similarly, there exist fixed ratios of splits of profits between writers and publishers, inventors and investors, actors and agents, etc.

Rates of various factors, such as possible risks, established reputation, possession of patents, etc., are established every day, and, at least for a specific group of players may be regarded as fixed. Perhaps, a systematic survey of these rates is of great interest.

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(1) This coalition cannot threaten the above outcome to a significant degree.

c. <u>Present negotiations</u>.

Even if very little is known prior to playing a game, one can at least
estimate the function $\varphi([P])$ by watching the negotiations which take place during
the playing of the game. There is, however, an inherent difficulty in this proce-
dure due to the fact that what we observe is a mixture of the "standard of fairness"
and the results from actual bargaining.

In the absence of any knowledge/the ·psychology of the players, their
feelings of justice and their bargaining abilities, and if the ·roles of the
various pressure groups, when cooperating to get the imputation, are of the same
nature, it is still of interest to provide for a "theoretical" standard of fairness.
This may serve as an a priori guidance to determine which pressure group should.form.
We shall indicate here two possible models.

d. <u>A cooperative standard of fairness</u>.

In this model, the players consider multi-person coalitions of pressure
groups, other than the grand coalition, as completely irrelevant. I.e., the pressure
groups which are formed, enter the final imputation with "equal rights." From this
point of view, we define $\varphi([P])$ to be:

$$(3.3) \quad \varphi_j([P]) = v(P_j) + \frac{v(N) - v(P_1) - v(P_2) - \ldots - v(P_m)}{m}, \quad j = 1, 2, \ldots, m,$$

for each partition $[P] \equiv (P_1, P_2, \ldots, P_m)$.

This is the "fair share" principle suggested in Section 2.

e. <u>A standard of fairness based on Shapley value</u>.

Let $(v;N)$ be a cooperative n-person game. In this model, the players
regard the partition $[P]$ as a final partition chosen by <u>explicit consent</u> of the
players in each pressure group, but the negotiations continue with possible formations
of coalitions of pressure groups. The pressure groups, however, do not know which of
the possible coalitions will result. More precisely, the pressure groups P_1, P_2, $\ldots$, P_m
in a partition $[P]$ consider themselves playing in an m-person game $(v^*;[P])$, having the
characteristic function

$$(3.4) \qquad v^*(P_{j1}, P_{j2}, \ldots, P_{jk}) = v(P_{j1} \cup P_{j2} \cup \ldots \cup P_{jk}).$$

We said, previously, that the original game should be regarded as a pseudo game among the various pressure groups, who may appear of disappear. Therefore, it is important for each pressure group, in each partition [P] in which it may be a member, to evaluate its prospects when playing the game.

We advocate that such an evaluation should be the Shapley value (see [8]) for the game $(v^*;[P])$; i.e.:

$$(3.5) \qquad \varphi_j([P]) = \sum_S \frac{(s-1)! \ (m-s)!}{m!} [v^*(S) - v^*(S - P_j)], \quad j = 1, 2, \ldots, m,$$

where m is the number of the pressure groups in [P], the S's are all possible coalitions of the pressure groups and s is the number of pressure groups in a coalition S.[1]

For a justification of this choice, we refer the reader to L. Shapley [8] and to R. D. Luce and H. Raiffa [3], where the merits of this value both as an a priori estimation and as an arbitration scheme are discussed. For further applications of Shapley value, see L. S. Shapley and M. Shubik [10], I. Mann and L. S. Shapley [4].

In L. S. Shapley and M. Shubik [10], an interesting experimental method is indicated to measure an empirical index for each player, which is based on ideas derived from the Shapley value. Contrary to the value, which does not take into account any of the sociological, psychological and other structures among the players, this empirical index attempts to take such factors into account simply by observing previous records of formations of coalitions among the players. Inasmuch as the power of a player (= pressure group) is a consequence of his and the other players actions in the past, this empirical index may give a good estimation of a standard of fairness.[2]

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(1) Note that (3.3) is Shapley value to the game $(u;[P])$ described in (3.1) and (3.2).

(2) The empirical index was applied in [10] to voting games, but it can be utilized also in other situations.

4. The power of a coalition.

Let $(\varphi([P]);N)$ be a game space, which results from a game $(v;N)$. Here v satisfies (2.1) and (2.2.), and $\varphi([P]) \equiv (\varphi_1[P], \varphi_2([P]), \ldots, \varphi_m([P]))$ satisfies (2.7) and (2.8), for each partition $[P] \equiv (P_1, P_2, \ldots, P_m)$ of N.

If a coalition B_j is formed in an intermediate stage, it can decide to partition itself into pressure groups in various ways. Suppose that a coalition structure[1] $(B_1, B_2, \ldots, B_t)$ is formed, and that each coalition has decided on its pressure groups, then a partition $[P]$ of N into pressure groups results.

We shall say that in this situation, the coalition B_j receives a payoff

$$(4.1) \qquad \sum_{P_\upsilon \subset B_j} \varphi_\upsilon ([P]), \quad j = 1, 2, \ldots, t.$$

Thus, we now conceive the original game as a pseudo game between all possible coalitions who may or may not form, given in normal form. The pure strategies for each coalition are the partitions of itself into pressure groups. This pseudo game is constant-sum, because the sum of the payments to the coalitions is, by (2.8), always $v(N)$.

If a coalition B is formed, then the larger the payoff it will receive, the smaller will the total payoff to the rest of the players be. Thus, the interests of B and $N-B$ are strictly opposing. Moreover, any strategy which is available to a partition of $N-B$ into coalitions is also available to the coalition $N-B$. Since side payments among the members of $N-B$ are permitted, we expect that if B forms, then the players in $N-B$ will pick the best strategy which is available to the coalition $N-B$, because for any other strategy, this strategy with appropriate profit splits will be preferred by each one of the players in $N-B$. All this amounts to saying that for each coalition B there exists a "natural" 2-person constant-sum game between B and $N-B$.

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(1) I.e., a partition of N into coalitions.

<u>Definition 4.1</u> Let $(\varphi([P]);N)$ be a game space. The <u>power</u> $w(B)$ of a coalition

B is the von Neumann minimax value (see [12]) of the 2-person game between B and

N-B, where the pure strategies for B and N-B are the partitioning of these coali-

tions into pressure groups and where the payoffs are defined by (4.1).

<u>Definition 4.2</u> Let $(\varphi([P]);N)$ be a game space, then the game $(w;N)$ will be called

the <u>derived game</u>.

<u>Theorem 4.1</u> <u>The derived game is a super-additive constant-sum game.</u>

<u>Proof</u>: By the minimax theorem (see J. von Neumann and O. Morgenstern [12]),

$$(4.2) \qquad\qquad w(B) + w(N - B) = v(N)$$

for each non-empty coalitions B and N-B. The same formula holds if either B or

N-B is the empty coalition, since an empty coalition always receives zero payoff

and $w(N) = v(N)$. If B and C are disjoint coalitions, then their union can be

sure of having a power of at least $w(B) + w(C)$, because it can use any product of

mixed strategies of B and C.

So far we made very little use of the original characteristic function of

the game. We merely operated within a game given by Thrall generalized characteristic

function $\varphi([P])$, which satisfies an additional assumption:

$$(4.3) \qquad\qquad \sum_{j=1}^{m} \varphi_j([P]) = \text{constant}$$

for each partition $[P] \equiv (P_1, P_2, \ldots, P_m)$ of N.

If the game is given a priori in this way, and if the pure strategies for

each coalition B are the partitioning of B into pressure groups, then the power

$w(B)$ is precisely the von Neumann-Morgenstern characteristic function of this game.

We took, however, the view that the game was given originally in terms of a charac-

teristic function v, and that the function $\varphi([P])$ was depending on v. The follow-

ing two questions therefore arise:

<u>a</u>. Are there relations between $(v;N)$ and $(w;N)$?

<u>b</u>. The function $\varphi([P])$ depends in general on v. On the basis of this function,

we claim that the coalitions realize that their actual "power" is given by w. To

this w they may now attribute a new and different function $\varphi^*([P])$ and iterate

the same procedure arriving perhaps at a different game (w*;N). This process may

perhaps be carried on indefinitely, continually yielding different powers. If this

is the case, are we right in basing our arguments on $\varphi([P])$ in the first place?

These questions will be treated in the remaining part of this section.

Theorem 4.2 Let $(\varphi([P]);N)$ be a game space which results from a game (v;N);

then, the power of a coalition is never smaller than its value.

Proof: This is trivially true for the coalition N and the empty coalition. Any other

coalition B can act as a one pressure group and thus assures itself the amount

Min $\varphi_1([P])$, where [P] runs over all the partitions of N which contain B as their

first element. By (2.7), this amount is greater than or equal to v(B). Therefore

$w(B) \geq v(B)$.

Theorem 4.3 Let $(\varphi([P]);N)$ be a game space, which results from a game (v;N). If

a partition $(B_1, B_2, ..., B_t)$ of N satisfies

(4.4) $$v(B_1) + v(B_2) + ... + v(B_t) = v(N),$$

then $$w(B_j) = v(B_j) \text{ for each } j, \quad j = 1, 2, ..., t.$$

Proof: By Theorem 4.1, it follows that

(4.5) $$w(B_1) + w(B_2) + ... + w(B_t) \leq v(N),$$

and by Theorem 4.2, the left hand side of (4.5) is greater than v(N), if a coalition

B_j exists for which $w(B_j) > v(B_j)$. This is impossible, hence $w(B_j) = v(B_j)$ for

each coalition B_j, $j = 1, 2, ..., t$.

Corollary 4.1 The derived game of a constant-sum game is identical to the game itself;

i.e., $w(B) = v(B)$ for each coalition B.

This corollary answers the criticism raised in b. above. A second derivation

of the game will not lead to a new game, no matter what the new function $\varphi^*([P])$ may

be. We feel that this feature is a great support to the theory.

5. Thrall power of a coalition.

It remains to question whether it is realistic for a coalition to determine its partition into pressure groups by "flipping a coin," for the minimax value may call for such a strategy.

We feel that in many cases this is not done, either because the players do not know better, or because it is not sound "to do business this way." Of course, this raises the delicate question as to who will be the first coalition to announce its choice of pressure groups. We shall not push this line of thought further, although it certainly deserves a thorough study. However, we feel that it is appropriate to offer a measure of power based on the assumption that mixed strategies are not available. For such situations we take R. M. Thrall's approach (see [11]), and offer the maxmin in pure strategies as a measure of power for each coalition. This definition is not so elegant and its consequences are less satisfactory compared to the previous definition, yet it seems to be more realistic in many situations. We shall discuss this concept here and show some interesting relations between the two power concepts in Section 8.

Definition 5.1 Let $(\varphi([P]);N)$ be a game space. The *Thrall power* $w^*(B)$ of a coalition B is

$$(5.1) \qquad w^*(B) = \underset{[B]}{\text{Max}} \ \underset{[P]}{\text{Min}} \ \underset{B_j \in [B]}{\Sigma} \ \varphi_j([P]),$$

where $[B]$ runs over all the possible partitions of B into pressure groups and $[P]$ runs over all the partitions of N into pressure groups which contain the elements of the given partition $[B]$ as their elements.

Definition 5.2 Let $(\varphi([P]);N)$ be a game space, then the game $(w^*;N)$ will be called the *Thrall derived game*.

Theorem 5.1 (R. M. Thrall [11]). *The Thrall derived game has a super-additive characteristic function.*

Theorem 5.2 *The Thrall power of a coalition is not smaller than its value.*

Proof: Same as the proof of Theorem 4.2.

<u>Theorem 5.3</u> <u>Let</u> ,(φ([P]);N) <u>be a game space which results from a game</u> (v;N). <u>If</u>

<u>a partition</u> (B$_1$, B$_2$, ...,B$_t$) <u>of</u> N <u>satisfies</u>

(5.2 $v(B_1) + v(B_2) + ... + v(B_t) = v(N)$,

<u>then</u> $w^*(B_j) = v(B_j)$ <u>for each</u> j,   j = 1, 2, ..., t.

<u>Proof</u>: Same as the proof of Theorem 4.3.

<u>Corollary 5.1</u> <u>The Thrall derived game of a constant-sum game is identical to the game</u>

<u>itself</u>.

Obviously, the power of a coalition B is equal to Thrall power, whenever

the payoff matrix of the game between B and N-B has a saddle point. This always

occurs if B contains 1 or n-1 players. Also $w^*(\phi) = w(\phi) = 0$, $w^*(N) = w(N)$.

Therefore, the derived game and Thrall's derived game are identical for n-person

games with $n \leq 3$. In general, however, $w^*(B) \leq w(B)$, and simple examples for each

n, $n \geq 4$, can be given, where strict inequality occurs.

Since the Thrall derived game may be non-constant sum, one ought to study

further iterations of the derivation process, in which the players realize each time

that the φ([P]) function should be "corrected." Let $w^*(B)$, $w^{**}(B)$. ... be a

sequence of such iterations, then, by Theorem 5.2. it is monotonically increasing.

It is also a bounded sequence, bounded, say, by v(N), therefore it must converge.

We have thus proved:

<u>Theorem 5.4</u> <u>An infinite sequence of iterations of Thrall derivations of a game always</u>

<u>converges, no matter what rule is chosen to derive the corresponding</u> φ([P]) <u>functions</u>

<u>from the corresponding</u> w^* <u>functions</u>.[1]

6. <u>The bargaining set</u> $\mathcal{M}_1^{(im)}$ <u>for a game space</u>.

Let (φ([P]);N) be a game space. We maintain that in deciding upon the

possible outcomes, the players treat the power w(B) of the various coalitions as

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[1] Provided, of course, that the analogues of (2.7) and (2.8) hold.

their basis for negotiations.

Since $(w;N)$ is a super-additive game, von Neumann-Morgenstern solution theory [12] can immediately be applied. The bargaining set theory (see R. J. Aumann and M. Maschler [1]) needs some modifications, due to the fact that the present analysis assumes a priori that an imputation should arise.

For the sake of completeness, we shall develop here briefly a bargaining-set analogous to the bargaining set $\mathcal{M}_1^{(1)}$ (defined implicitly in [1] and explicitly in [6] and in M. Davis and M. Maschler [2].

<u>Definition 6.1</u> Let $(w;N)$ be a derived game. A <u>group rational payoff configuration</u> (g.r.p.c.) is a pair $(x;\mathcal{B})$, where $x \equiv (x_1, x_2, \ldots, x_n)$ is an imputation, $\mathcal{B} \equiv (B_1, B_2, \ldots, B_t)$ is a coalition structure,[1] and $\mathcal{B}$ and x satisfy:

(6.1)
$$w(B_1) + w(B_2) + \ldots + w(B_t) = v(N),$$

(6.2)
$$\sum_{i \in B_j} x_i = w(B_j), \quad j = 1, 2, \ldots, t.$$

<u>Definition 6.2</u> Let $(x;\mathcal{B})$ be a g.r.p.c. for a derived game $(w;N)$, and let k and ℓ be two distinct players in a coalition B_j of $\mathcal{B}$. An <u>objection</u> of player k against player ℓ in $(x;\mathcal{B})$ is a pair $(\hat{y};C) \equiv (\{y_i\}_{i \in C};C)$, where C is a coalition which contains player k but does not contain player ℓ and where the payoff $\hat{y}$ satisfies:

(6.3)
$$\sum_{i \in C} y_i = w(C),$$

(6.4)
$$y_k > x_k,$$

(6.5)
$$y_i \geq x_i, \quad i \in C.$$

<u>Definition 6.3</u> Let $(x;\mathcal{B})$ be a g.r.p.c. for a derived game $(w;N)$, and let $(y;C)$ be an objection of a player k against a player ℓ in $(x;\mathcal{B})$. A <u>counter</u>

---

(1) I.e., a partition of N into mutually disjoint non-empty coalitions whose union is N.

<u>objection</u> to this objection is a pair $(\hat{\jmath};D) \equiv (\{z_i\}_{i \in D};D)$, where D is a coalition which contains player ℓ but does not contain player k, and where the payoff $\hat{\jmath}$ satisfies:

(6.6)
$$\sum_{i \in D} z_i = w(D),$$

(6.7)
$$z_i \geq x_i, \ i \in D,$$

(6.8)
$$z_i \geq y_i, \ i \in D \cap C.$$

<u>Definition 6.4</u> Let $(w;N)$ be a derived game. A g.r.p.c. is said to belong to the <u>bargaining set</u> $\mathcal{M}_1^{(im)}$, if any objection of a player k against a player ℓ has a counter objection.

<u>Remark 6.1</u> One observes easily that $\mathcal{M}_1^{(im)}$ contains exactly those elements of the bargaining set $\mathcal{M}_1^{(i)}$, for which the coalition structure $\mathcal{B}$ satisfies (6.1).

<u>Remark 6.2</u> The situation is more complicated if Thrall power is used. In such cases, it may happen that a coalition C cannot be an element of a coalition structure which satisfies (6.1). One may argue that it cannot be used in an objection, since the players "prohibit" non-imputations outcomes. A similar argument holds for the freedom in making counter objections.

For further results concerning $\mathcal{M}_1^{(i)}$ (and therefore, also $\mathcal{M}_1^{(im)}$, we refer the reader to M. Davis and M. Maschler [2] and to B. Peleg [7]. In particular, it follows immediately from B. Peleg [7], that to each coalition structure satisfying (6.1), there exists a payoff X, such that $(X;\mathcal{B}) \in \mathcal{M}_1^{(im)}$.

7. <u>The three person game.</u>

For a 2-person game space, there can be only one outcome. The interest in computing the power of a coalition starts with the 3-person games. In this section, we shall analyze such games for the cooperative standard of fairness and the standard of fairness based on Shapley value (see Section 3). As both standards yield results which are invariant under strategic equivalence, we shall assume that $v(i) = 0$, $i = 1, 2, 3$.

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a. The cooperative standard of fairness.

Applying formula (3.3) to Definition 4.1, one observes that the various powers take the form:

$$(7.1) \begin{cases} w(i) = \text{Min } (v(123)/3; \ [v(123) - v(jk)]/2), \ i, \ j, \ k \text{ distinct}, \ i = 1, 2, 3, \\ w(ij) = \text{Max } (2v(123)/3; \ [v(123) + v(ij)]/2), \ i, \ j = 1, 2, 3, \quad i \neq j, \\ w(123) = v(123). \end{cases}$$

Without loss of generality we may assume that $v(12) \geq v(23) \geq v(13)$, and then four cases may occur:

Case A. If $3v(13) \geq v(123)$, then the derived game is:

$$(7.2) \begin{cases} w(i) = [v(123) - v(jk)]/2, \ i = 1, 2, 3, \ i, \ j, \ k \text{ distinct}, \\ w(ij) = [v(123) + v(ij)]/2, \ i, \ j = 1, 2, \ i \neq j, \\ w(123) = v(123). \end{cases}$$

In this case, each 2-person coalition, if it forms, will prefer to act as a 1-person pressure group. The bargaining $\mathcal{M}_1^{(im)}$ set of this game is

$$(7.3)$$

$$\begin{cases} (\frac{v(123)+v(12)+v(13)-v(23)}{4}, \ \frac{v(123)+v(12)+v(23)-v(13)}{4}, \ \frac{v(123)-v(12)}{2} \ ;12, \ 3) \\ (\frac{v(123)+v(12)+v(13)-v(23)}{4}, \ \frac{v(123)-v(13)}{2}, \ \frac{v(123)+v(23)+v(13)-v(12)}{4} \ ;13, \ 2) \\ (\frac{v(123)-v(23)}{2}, \ \frac{v(123)+v(12)+v(23)-v(13)}{4}, \ \frac{v(123)+v(23)+v(13)-v(12)}{4} \ ;23, \ 1) \\ (\frac{2v(123)+v(12)+v(13)-2v(23)}{6}, \frac{2v(123)+v(12)+v(23)-2v(13)}{6}, \frac{2v(123)+v(23)+v(13)-2v(12)}{6} \ ;123), \end{cases}$$

and the payoff vectors in the first three payoff·configurations form the von Neumann-Morgenstern non-discriminating solution to this game. The discriminatory solutions can be read from these three imputations in an obvious way. Note that the last payoff is an average of the first three.

Everything that was written in the previous paragraph, after (7.3), is true for all the other cases,[1] since the derived games are always constant-sum games. Therefore, we shall only state the bargaining set in these cases, without repeating the rest of the description.

---

(1) Referring, of course, to other bargaining sets.

__Case B.__ If $3v(13) < v(123)$ and $3v(23) \geq v(123)$, then the derived game

takes the form:

$$
(7.4) \quad
\begin{cases}
w(i) & = [v(123) - v(jk)]/2, \quad i = 1, 3, \quad i, j, k \text{ distinct,} \\
w(2) & = v(123)/3, \\
w(ij) & = [v(123) + v(ij)]/2, \quad ij = 12, 23, \\
w(13) & = 2v(123)/3, \\
w(123) & = v(123).
\end{cases}
$$

This case differs from the previous one by the fact that coalition {13} will

prefer to act as two pressure groups, if it forms.

The bargaining set $\mathcal{M}_1^{(im)}$ for this game is:

(7.5)

$$
\begin{cases}
(v(123)/3+v(12)/4-v(23)/4, & v(123)/6+v(12)/4+v(23)/4, & v(123)/2-v(12)/2, & ; 12,3) \\
(v(123)/3+v(12)/4-v(23)/4, & v(123)/3, & v(123)/3+v(23)/4-v(12)/4, & ; 13,2) \\
(v(123)/2-v(23)/2, & v(123)/6+v(12)/4+v(23)/4, & v(123)/3+v(23)/4-v(12)/4, & ; 23,1) \\
\left(\dfrac{7v(123)+3v(12)-6v(23)}{18}, & \dfrac{4v(123)+3v(12)+3v(23)}{18}, & \dfrac{7v(123)+3v(23)-6v(12)}{18}, & ; 123\right).
\end{cases}
$$

__Case C.__ If $3v(23) < v(123)$ and $3v(12) > v(123)$, then each of the coalitions

{13} and {23}, if it forms, will prefer to act as two pressure groups. The derived game

will be:

$$
(7.6) \quad
\begin{cases}
w(1) & = w(2) = v(123)/3, \\
w(3) & = [v(123) - v(12)]/2, \\
w(12) & = [v(123) + v(12)]/2, \\
w(13) & = w(23) = 2v(123)/3, \\
w(123) & = v(123),
\end{cases}
$$

and the bargaining set $\mathcal{M}_1^{(im)}$ is:

$$(7.7) \begin{cases} ([v(123) + v(12)]/4, & [v(123) + v(12)]/4, & [v(123) - v(12)]/2 \quad ; 12,3) \\ ([v(123) + v(12)]/4, & v(123)/3, & [5v(123) - 3v(12)]/12 \; ; 13,2) \\ (\quad v(123)/3, & [v(123) + v(12)]/4, & [5v(123) - 3v(12)]/12 \; ; 23,1) \\ ([5v(123) + 3v(12)]/18, & [5v(123) + 3v(12)]/18, & [4v(123) - 3v(12)]/9 \quad ; 123). \end{cases}$$

__Case D.__ If $3v(12) \leq v(123)$, then no 2-person coalition, if it forms, will prefer to act as a one pressure group. The derived game will be inessential:

$$(7.8) \begin{cases} w(i) & = v(123)/3, \quad i = 1, 2, 3, \\ w(ij) & = 2v(123)/3, \quad i,j = 1, 2, 3, \quad i \neq j, \\ w(123) & = v(123), \end{cases}$$

and the bargaining set will assign each player the amount $v(123)/3$, in each coalition structure.

__b.__ __The standard of fairness based on Shapley value.__

Applying (3.5) to Definition (4.1), one easily finds that the powers of the coalitions in a 3-person game are:

$$(7.9) \begin{cases} w(i) & = \text{Min } \{[v(123) - v(jk)]/3 + [v(ij) + v(ik)]/6, [v(123) - v(jk)]/2\}, \\ & \qquad\qquad i = 1, 2, 3, \qquad i, j, k \text{ distinct}, \\ w(ij) & = \text{Max } \{[2v(123)]/3 + v(ij)/3 - [v(ik) + v(jk)]/6, [v(123) + v(ij)]/2\}, \\ & \qquad\qquad i, j = 1, 2, 3, \quad . \quad i, j, k \text{ distinct}, \\ w(123) & = v(123). \end{cases}$$

Two cases are to be considered:

__Case A.__ If $v(123) \geq v(12) + v(13) + v(23)$, then

$$(7.10) \begin{cases} w(i) & = [v(123) - v(jk)]/3 + [v(ij) + v(ik)]/6, \; i = 1, 2, 3, \quad i, j, k \text{ distinct}, \\ w(ij) & = w(i) + w(j) = 2v(123)/3 + v(ij)/3 - [v(ik) + v(jk)]/6, \\ & \qquad\qquad i, j = 1, 2, 3, \quad i, j, k \text{ distinct}, \\ w(123) & = w(1) + w(2) + w(3) = v(123). \end{cases}$$

This is an inessential game, since each 2-person coalition, should it form, is better off by playing as two pressure groups. The von Neumann-Morgenstern solution of this game consists of one imputation which assigns player i, $i = 1, 2, 3$, the amount $w(i)$, which is the Shapley value of the original game. The same imputation appears also in $\mathcal{M}_1^{(im)}$ for each coalition structure.

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Case B. If $v(123) < v(12) + v(13) + v(23)$, then _each_ 2-person coalition will prefer to act as as one pressure group, in case it forms. In this case, the situation is the same as in case A of the cooperative standard of fairness. The derived game is given by (7.2), the bargaining set $\mathcal{M}_1^{(im)}$ is (7.3), and von Neumann-Morgenstern solutions are derived the same way as there.

Notice that the imputation in the last payoff configuration of (7.3) is exactly Shapley value for the original game.

In both cases, A and B, the game and the derived game have the same Shapley value. This follows from the fact that Shapley value always appears in the payoff configuration of $\mathcal{M}_1^{(im)}$ having (123) as a coalition structure, and from the fact that a derivation of the derived game yields back the derived game (Theorem 4.1 and Corollary 4.1).

To get some idea of the "flavour" of the present theory, we shall conclude the section with a detailed discussion of a particular game, similar to the one given in the introduction and to the one described in Example 2.2, in the sense that one coalition in it appears to be particularly weak.
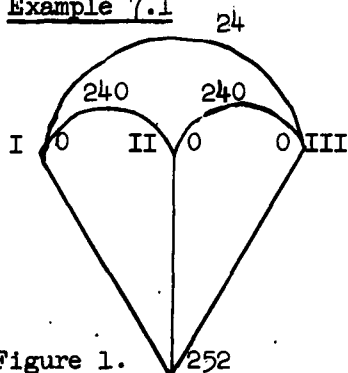
Example 7.1

Figure 1.

Figure 2.

Consider the 3-person game, described by Figure 1. (The players are denoted by Roman numerals). This game has a one-point core (12, 228, 12), which is the "center" of each von Neumann-Morgenstern solution[1]. Also, the only point in the bargaining set $\mathcal{M}_1^{(im)}$ for this game is (12, 228, 12; 123). The Shapley·value for this game, however, is (48, 156, 48). Under the cooperative standard of fairness, this game is transformed into the constant-sum game given by Figure 2. The derived game has an empty core. Its bargaining set $\mathcal{M}_1^{(im)}$ is given by
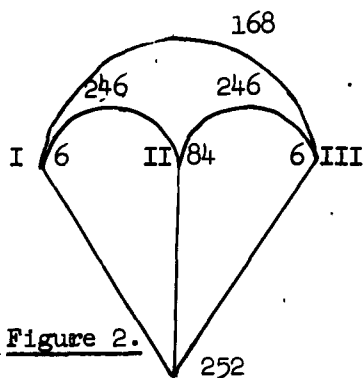
_____
(1) Each solution is obtained by adding three "wiggles" issuing from this point.

$$(7.11) \quad \left\{ \begin{array}{l} (84, \ 162, \ \ 6; \ 12, \ 3) \\ (84, \ \ 84, \ 84; \ 13, \ 2) \\ (\ 6, \ 162, \ 84; \ 23, \ 1) \\ (58, \ 136, \ 58; \ 123), \end{array} \right.$$

from which, von Neumann-Morgenstern solutions can easily be derived.

Discussion: The players know, by the standard of fairness, that the pressure groups will enter the 3-person enterprise with "equal rights," i.e., with excess profits equally divided by the pressure groups, and given in addition to the profits which each pressure group obtains by itself. Therefore, coalition {23} is not so weak as it first appeared, since it can threaten to first form and then act as two pressure groups. Practically this means that players 1 and 3 come to any agreement not to listen to any offer made by player 2 to one of them. A similar strategy is not profitable to the coalitions {12} and {23}. These coalitions would prefer to act as a 1-person pressure group, by letting one of their members/drop out of the game, for instance, after being promised a side payment, or by sending a single lawyer to represent the coalition which forms.

. The payoffs in (7.11) indicate what the side payments should be in case an intermediate coalition forms[1]. For instance, if {13} forms, they "should" split equally the 168 points that they can get, leaving player 2 with 84 points. The only way player 2 can avoid the formation of such a coalition is to try and form an intermediate coalition with either player 1 or player 3, in which case he has to offer his partner "about" 84 points as a side payment, which is the amount that this player would have received in the first case.

Thus, the first three payoff configurations are quite stable. Notice, however, that each of them is less preferred by one of the players. This player has something to say, for he can point out that without his cooperation, the other players will never realize their side payments, since their value is smaller than their power. Of course, this player also wants to profit and he realizes that an imputation should result,

_____

(1) This is not true in a general case, since there is no unique p.c. in the bargaining set for each coalition structure.

but why not arrange for an imputation without forming intermediate stages two person coalitions? The other players may be willing to consent to such an offer, especially since every one has some chance to be the one left out. In this case, they may settle at the third payoff configuration of (7.11), where each one agrees to "sacrifice" the same amount: $84 - 58 = 162 - 136 = 84 - 58 = 26$.

Thus, although none of these outcomes is in the core of the original game, they all exhibit stability features, provided the standard of fairness is indeed the accepted rule.

Under the standard of fairness based on Shapley value, the original game is transformed into a different constant-sum game, given in Figure 3. The bargaining set $\mathcal{M}_1^{(im)}$ for this game, from which von Neumann-Morgenstern solutions can be determined in a glance, is:
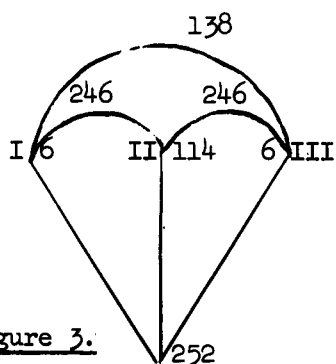
138
246   246
I 6   II 114   6 III

252

Figure 3.

(7.12)

$$\begin{cases} (69, 177, 6; 12, 3) \\ (69, 114, 69; 13, 2) \\ (6, 177, 69; 23, 1) \\ (48, 156, 48; 123). \end{cases}$$

Discussion: This time, the players do not expect "equal rights." Instead, each coalition evaluates a priori the various prospects of breaking into pressure groups. In this case, each 2-person coalition prefers to act as a one pressure group. (In particular, this is true for the coalition {13}, since if it acts as two pressure groups, it will expect only 96 which is the sum of their Shapley values).

The rest of the discussion is essentially the same as in the previous case and therefore will be omitted. Again, the core does not appear in the outcomes but this time Shapley value does.

8. <u>The derived game and Thrall's derived game</u>.

In this section we shall prove the following curious theorem:

<u>Theorem 8.1</u> <u>If the standard of fairness is of the cooperative type, then the 2-person games which determine the powers of the various coalitions can be solved in pure strategies. In other words, the Thrall derived game and the derived game are identical.</u>

<u>Proof</u>: There is nothing to prove if B is the empty or the grand coalition. Let B be any other coalition, and let $\{a_{ij}\}$ be the payoff matrix of the game between B and N-B, where the pure strategies are the partitions of B and N-B into pressure groups and the payoffs are determined by (3.3) and by (4.1) (B replaces B_j). Suppose that this matrix has no saddle point.

The following assumptions are consequences of renaming, if necessary, the various strategies, and therefore entail no loss of generality:

<u>a</u>. $a_{11} = \underset{i}{Max}\ \underset{j}{Min}\ a_{ij}$,

<u>b</u>. $a_{11} < a_{21}$,

<u>c</u>. $a_{22} = \underset{j}{Min}\ a_{2j}$.

It follows that

(8.1) $$a_{11} \leq a_{12}, \quad a_{11} \geq a_{22}, \quad a_{11} < a_{21}.$$

Let $(B_1^1, B_2^1, \ldots, B_\alpha^1)$ and $(B_1^2, B_2^2, \ldots, B_\alpha^2)$ be the partitions of B into pressure groups, used by B in his first and second strategies, respectively. Let $(c_1^1, c_2^1, \ldots, c_\beta^1)$ and $(c_1^2, c_2^2, \ldots, c_\delta^2)$ be the partitions of N-B, into pressure groups, used by N-B in his first and second strategies, respectively. We denote:

(8.2)
$$\sum_{j=1}^{\alpha} v(B_j^1) = E, \quad \sum_{j=1}^{\beta} v(c_j^1) = F,$$

$$\sum_{j=1}^{\gamma} v(B_j^2) = G, \quad \sum_{j=1}^{\delta} v(c_j^2) = H.$$

By (3.3),

(8.3) $$a_{11} = E + \frac{\alpha}{\alpha+\beta} [v(N) - E - F],$$

(8.4) $$a_{12} = E + \frac{\alpha}{\alpha+\delta} [v(N) - E - H],$$

(8.5) $$a_{21} = G + \frac{\gamma}{\gamma+\beta} [v(N) - G - F],$$

(8.6) $$a_{22} = G + \frac{\gamma}{\gamma+\delta} [v(N) - G - H].$$

Substituting these expressions in (8.1), and simplifying, we obtain

(8.7) $(\delta - \beta) v(N) \leq (\delta - \beta) E + (\alpha + \delta) F - (\alpha + \beta) H,$

(8.8) $(\alpha - \gamma) v(N) < (\alpha + \beta) G + (\alpha - \gamma) F - (\beta + \gamma) E,$

(8.9) $(\alpha\delta - \beta\gamma) v(N) \geq \delta(\alpha + \beta) G + \alpha (\gamma + \delta) F - \beta(\gamma + \delta) E - \gamma(\alpha + \beta) H.$

Multiplying each side of (8.7) by γ and each side of (8.8) by δ and taking

the sum, we arrive at a contradiction to (8.9). This shows that the assumption that

$\{a_{ij}\}$ has no saddle point is wrong.

An analogous theorem, when Shapley value is the basis of the standard of

fairness, is true only for games with a number of players smaller than 5. This is

obvious if there are up to 3 players in the game, and we shall omit the proof for

4-person games. A counter example for 5-person games is the following:

Example 8.1 Consider the 5-person game whose characteristic function is

$v(12345) = 240$, $v(1234) = v(1245) = v(1235) = 204$, $v(2345) = v(1345) = 144$, $v(B) = 0$

otherwise. If the standard of fairness is based on Shapley value, then the derived

game is: $w(\phi) = 0$, $w(1) = w(2) = 32$, $w(3) = w(4) = w(5) = 12$, $w(12) = 111\frac{9}{11}$,

$w(13) = w(14) = w(15) = w(23) = w(24) = w(25) = 104$, $w(34) = w(35) = w(45) = 101$,

$w(B) = w(N-B)$.

The Thrall derived game has the same characteristic function, except that

$w(12) = 111$ and $w(345) = 126$.

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